Permutations and combinations

★ Fundamental Principle of counting

(i) Multiplication Principle

→ If an event has two stages or two operations.
First with m no. of ways
Second with n no. of ways

Event will occur Total $m \times n$ ways

Example

From city A to city B there are 3 ways
તો આવવા જવાના total ways
$3 \times 3 = 9$ એટલે કે 9 અલગ અલગ રીતે આવી કે જઈ શકાય.

(ii) Addition Principle

જો સવારે તમારી સાથે જવાવાળા ૩ છે
સાંજે જવાવાળા ૩ છે
તો Total તમારી સાથે દિવસમાં જવાવાળા લોકો $= 3 + 3 = 6$

Permutation:- Every of Different arrangements which can be made by taking some or all of a number of things is called a Permutation

Eg. A, B, C → ABC, ACB, BCA, BAC, CBA, CAC
AB, BA, AC, CA, BC, CB, A, B, C.

Factorial notation

$5 \times 4 \times 3 \times 2 \times 1 = \lfloor 5$ or $5!$

$\lfloor n = n! = n(n-1).(n-2) \ldots 3 \times 2 \times 1$

$\lfloor n = n\,(n-1)\,(n-2)\,\lfloor (n-3)$

$n = +$ive integer

Some imp Prop

$\lfloor n = n \lfloor n-1$, $\lfloor 2n = 2^n \lfloor n\,[1 \cdot 3 \cdot 5 \ldots (2n-1)]$

$\lfloor 1 = \lfloor 0 = 1$, $\lfloor n-1 = (n-1)(n-2) \lfloor (n-3)$

${}^nP_r \Rightarrow$ No. of ways in which r things taken at a time from n different things and the Permutation is done of r things.

$(n > r)$

n thing માંથી r thing ને ગોઠવવાની હોય તો

Permutation $^nP_r = \frac{\lfloor n}{\lfloor n-r}$

Ex 16 Player માંથી 11 Player ને લેવાના અને તો

$\frac{\lfloor 16}{\lfloor 16-11}$ } તેને Batting order માં ગોઠવવા માટે $\lfloor 11$ રીતે ગોઠવી શકાય.

$\lfloor n$ = Permutation of n different things in a row

$$^nP_r = {}^nC_r \times \lfloor r = \frac{\lfloor n}{\lfloor n-r}$$

Imp

The no. of Permutation of n diff. things taken r at a time when each thing may be Reapeated any no. of time $= n^r$

Ex. A, B, C, D, E, F, G } 7 અક્ષરને 4 લેવાના અને ગોઠવવા હોય તો $\boxed{7}\boxed{7}\boxed{7}\boxed{7}$ બધી જગ્યાએ 7 Choice

$\therefore 7 \times 7 \times 7 \times 7 = 7^4$ રીતે {કોઈ no. Repeat કરી શકાય છે. તેથી આમ થાય

Prop. ⇒ $^nP_0 = \frac{\lfloor n}{\lfloor n-0} = \boxed{1}$, $^nP_1 = \frac{\lfloor n}{\lfloor n-1} = \boxed{n}$

Prove that ${}^{n}P_{r} = {}^{n-1}P_{r} + r \times {}^{n-1}P_{r-1}$ $(n > r)$

$$= \frac{\lfloor n-1}{\lfloor n-1-r} + \frac{r \lfloor n-1}{\lfloor n-1-r+1}$$

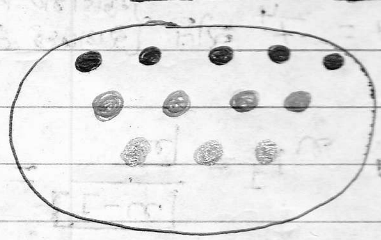
$$= \frac{\lfloor n-1}{\lfloor n-1-r} + \frac{r \cdot \lfloor n-1}{\lfloor n-r}$$

$$= \frac{(n-r) \lfloor n-1}{(n-r) \lfloor n-1-r} + \frac{r \lfloor n-1}{\lfloor n-r}$$

$$= \frac{(n-r) \lfloor n-1}{\lfloor n-r} + \frac{r \lfloor n-1}{\lfloor n-r}$$

$$= \frac{\lfloor n-1}{\lfloor n-r}\left(n - \cancel{r} + \cancel{r}\right] = \frac{n \lfloor n-1}{\lfloor n-r}$$

$$= \frac{\lfloor n}{\lfloor n-r} = \boxed{{}^{n}P_{r}}$$



p = 5 Blue Balls

q = 4 Red Balls

r = 3 Green Balls

$n = 5+4+3 = 12$

taken all at a time

$$\frac{\lfloor n}{\lfloor p \lfloor q \lfloor r} = \frac{\lfloor 12}{\lfloor 5 \ \lfloor 4 \ \lfloor 3}$$

The no. of arrangements of n things taken all at a time, p are alike, q are alike, r are alike and the Rest $n-(p+q+r)$ are all different is $\lfloor n / \lfloor p \lfloor q \lfloor r$

Example Find all arrangement of letters ~~of word~~ MISSISIPI

$n = 9$, $p = 4$, $q = 3$ $= \dfrac{\lfloor\underline{9}}{\lfloor\underline{4}\ \lfloor\underline{3}}$

⇒ No. of Permutation with certain condition

(i) n different thing, r taken at a time one Particular thing is taken every time

$$= r \cdot {}^{n-1}P_{r-1}$$

⇒ When a Particular thing is never taken

n વસ્તુઓમાંથી કોઈ દિવસ એક વસ્તુ લેવાની જ નથી. તેથી તેના માટે $n-1$ વસ્તુ જ હશે. કા.કે n માંથી જે નથી લેવાની તેને બાદ કરી તેને r વસ્તુ માં લેવામાં આવે તો $\boxed{{}^{n-1}P_r}$

⇒ No. of Permutation of n different things taken all at a time when m special things always come together is $m < n$

n different things છે, તેથી $\lfloor\underline{n}$ થાય પણ તેમાંથી m હંમેશા ભેગી તેથી $\lfloor\underline{n-m}$ થાય. હવે તેને ગોઠવતા $n-m+1$ કારણકે × × ● × × આવતી રીતે n માં 4 તો m ક્યાય તેથી $(n-m+1)$ અને આ અલગ m વાળી વસ્તુ $\lfloor\underline{m}$ અલગ અલગ રીતે ગોઠવાય.

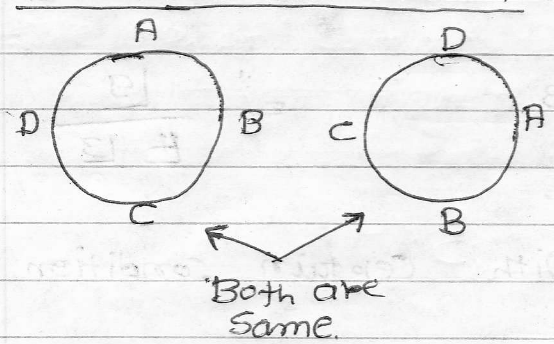
$$= \lfloor\underline{m}\ (n-m+1)\ \lfloor\underline{n-m}$$

$$\therefore \boxed{\lfloor\underline{m}\ \lfloor\underline{n-m+1}}$$

અહીં જો m ભેગી સાથે ન હોય તો અને $m-1$ હોય તો

$\lfloor\underline{n} - \lfloor\underline{n-m+1}\ \lfloor\underline{m}$

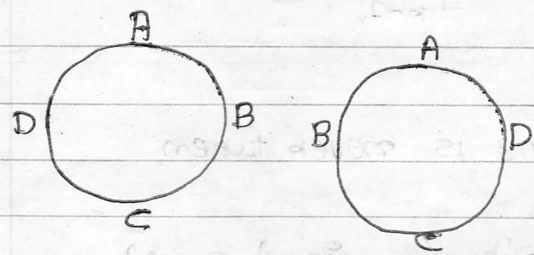
Circular Permutation



n different thing માટે

$$\frac{\lfloor n}{n} = \lfloor n-1$$

In this case clockwise and anticlockwise arrange are different.



આગળની જેમ એક તરફ કે બીજી તરફ તો same જ હોય. તેમાં કંઈ ફેર ન હોય.

$$= \frac{\lfloor n-1}{2}$$

⇒ No. of circular permutation when r things taken from n different things and clockwise and anticlockwise arragements are different

$$\frac{{}^nP_r}{r}$$

⇒ If clockwise and anticlockwire arragements are Same

$$\frac{{}^nP_r}{2r}$$

Combination

The no. of combinations of n different things taken r at a time is denoted as

$${}^{n}C_{r} \quad \boxed{or} \quad C(n,r) \quad \boxed{or} \quad \binom{n}{r}$$

$${}^{n}C_{r} = \frac{\lfloor n}{\lfloor n-r \; \lfloor r}$$

Example: 25 cricket Player માંથી 11 ની ટીમ બનાવવાની રીતો

$${}^{25}C_{11} = \frac{\lfloor 25}{\lfloor 11 \; \lfloor 14}$$

$${}^{n}C_{r} = \frac{\lfloor n}{\lfloor n-r \; \lfloor r} = \frac{{}^{n}P_{r}}{\lfloor r}$$

Properties

1. ${}^{n}C_{r}$ is a natural no. 1, 2, ...
2. ${}^{n}C_{0} = 1 = {}^{n}C_{n}$, ${}^{n}C_{1} = n$
3. ${}^{n}C_{r} = {}^{n}C_{n-r}$
4. ${}^{n}C_{r} + {}^{n}C_{r-1} = {}^{n+1}C_{r}$
5. If ${}^{n}C_{x} = {}^{n}C_{y} \Rightarrow x = y$ or $x + y = n$
6. If n is even greatest value of ${}^{n}C_{r}$ is ${}^{n}C_{n/2}$ i.e. $r = n/2$
7. If n is odd greatest value of ${}^{n}C_{r}$ is ${}^{n}C_{\left(\frac{n+1}{2}\right)}$ or ${}^{n}C_{\left(\frac{n-1}{2}\right)}$
8. ${}^{n}C_{r} = \frac{n}{r} \cdot {}^{n-1}C_{r-1}$

⇒ The no. of combination of n different thing taken r at a time. when k Particular objects always occur $\quad k < r < n$

$$^{n-k}C_{r-k}$$

⇒ If k Particular objects (things) never taken

$$^{n-k}C_{r}$$

⇒ The no. of ways of n different objects selecting at least one of them

Total no. of combinations of n diff. things $\rightarrow$ $^nC_1 + {}^nC_2 + {}^nC_3 + \dots + {}^nC_n = 2^n - 1$

$$\Rightarrow \quad {}^nC_0 + {}^nC_1 + {}^nC_2 + \dots + {}^nC_n = 2^n$$

If out of n their are (p, q, r, s) thing are such that P are a like, q are alike, r are alike and s are alike and $(n-p-q-r-s = t)$ and remaining t thing are different to each other then total no. of combination will be

$$(p+1)(q+1)(r+1)(s+1)2^t - 1$$

Example

$$n = 15$$

4 Red
5 yellow
3 White
1 black
1 blue
1 brown
each

$$(4+1)(5+1)(3+1)2^3 - 1$$

Division into groups :-

Total no. of ways in which $(m+n)$ different thing can be divided into two groups one group will contain m things and other will contain n thing respectively is

$$^{m+n}C_m \times {}^{n}C_n = {}^{m+n}C_m = \frac{\underline{|m+n}}{\underline{|m}\ \underline{|n}}$$

If order is not important and $2n$ diff. things is to be divided in group of n each

$$2n \longrightarrow n\ n = \frac{\underline{|2n}}{\underline{|2}\ \underline{|n}\ \underline{|n}}$$

ઓર્ડર important ન હોય તો $\underline{|2}$ થી ભાગવું
3 માટે $\underline{|3}$ થી, 4 માટે $\underline{|4}$ થી.

જો order important હોય તો

$$4n \rightarrow n\ n\ n\ n \quad \text{માટે} \quad \frac{\underline{|4n}}{\underline{|n}\ \underline{|n}\ \underline{|n}\ \underline{|n}}$$

$\underline{|4}$ થી divide ન કરવું

Ex. $m+n+p \Rightarrow m\ \ n\ \ p \Rightarrow \dfrac{\underline{|m+n+p}}{\underline{|m}\ \underline{|n}\ \underline{|p}\ \ \underline{|3}}$ order not imp.

Arrangements in group

The no. of arrangement, in which n diff. things can be arranged into r diff. groups is

$$\underline{|n} \times {}^{n-1}C_{r-1}$$

Ex. 5 dift. balls arranged in three diff. box

$n = 5$
$r = 3$

હવે ત્રણ બોક્સ બનાવવા ચાર અલગ અલગ જગ્યાએથી
બે line ખેંચવી પડશે. તેનાથી ત્રણ અલગ અલગ
બોક્સ બનશે. આ બોક્સ 4 લાઈનમાંથી 2ને ખેંચવાથી
બને તેથી

$$\lfloor 5 \times {}^4C_2$$